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On Vlasov–Manev Equations, II: Local Existence and Uniqueness

R. Illner,[1] H. D. Victory,[2] P. Dukes,[1] and A. V. Bobylev[1]

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We prove that the initial value problem associated with the Vlasov–Manev system (a Vlasov equation in which a correction of type ε/r^2 is added to the Newtonian or Coulomb potential) has a local in time classical and unique solution for sufficiently regular initial data.

KEY WORDS: Vlasov equation with Manev correction.

1. INTRODUCTION

This is the second in a series of papers in which we discuss kinetic equations where a correction to the Coulomb potential of type ε/r^2 is added; in part I [BDIV], we referred to such corrections as corrections of Manev type, in reference to the Bulgarian physicist Manev who studied such potentials [Ma1-4] in the 1920s. We are in particular interested in a generalization of the stellar dynamic equation of the type

$$\partial_t f + v \cdot \nabla_x f + (E_1[\rho] + E_2[\rho]) \cdot \nabla_v f = 0 \tag{1.1}$$

with $\rho = \int f\,dv$, and

$$E_1[\rho](t, x) = -\gamma(N-2) \int_{\mathbb{R}^N} \frac{x-y}{|x-y|^N}\, \rho(t, y)\, dy \tag{1.2}$$

$$E_2[\rho](t, x) = -\varepsilon \int_{\mathbb{R}^N} |x-y|^{-N+1}\, \nabla_y \rho(t, y)\, dy \tag{1.3}$$

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada.
[2] Department of Mathematics, Texas Tech University, Lubbock, Texas.

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Here, N is the dimension of the space (in part I we used $N=3$ throughout); by a formal partial integration, E_2 can also be written as a multiple of the Riesz transform of ρ.

Let $f_0(x, v)$ be a sufficiently regular density distribution function. Our objective is to prove that the initial value problem associated with (1.1) then has a local classical and unique solution. We mention here that our proof applies to the stellar dynamic case as well as the cases where the forces are repulsive, or where one of the forces given by (1.2) and (1.3) is repulsive and the other is attractive (i.e., where one or both of the minus signs on the right are changed to plus signs). These cases may be of some interest in plasma physics or electron flow in semiconductors.

We mention here that if $\varepsilon = 0$, (1.1-2) becomes the classical stellar dynamic equation, which was solved locally in time by R. Kurth in 1952 [K] and globally in the radially symmetric case by Batt in 1977 [B]. Since then, general global solvability has been established; see the references in [BDIV]. As we showed in BDIV, general global solvability does not apply when $\varepsilon > 0$.

In Section 2, we present the estimates on the fields E_1 and E_2 from (1.2) and (1.3) which enable us to prove a local existence result. Section 3 contains estimates on solutions of approximating mollified problems, in which the singularities in (1.2) and (1.3) are mollified. In Section 4, we send the mollification parameter δ to zero and use compactness properties to extract a local solution of our initial value problem. Finally, in Section 5, we show that the solution is unique if the initial data have enough regularity.

The key observation in the analysis is that the Manev potential does not improve on the regularity of the Newtonian potential, obtained by convolving with the classical Poisson kernel, but does not compromise it either. The nature of the singularity in the Manev correction means that we must seek solutions whose densities possess Hölder continuous spatial gradients, and this is the main novelty arising in the analysis. The most technical feature of the existence analysis is to show that Hölder continuity of the distribution and its density is maintained at least locally in time with no deterioration in the Hölder exponents. For this, the Hölder continuity of the particle trajectories with respect to initial states must be carefully studied.

In the appendix we sketch proofs of some known but crucial estimates on which our analysis depends. These estimates are listed in the following section.

2. SOME ESTIMATES

Let $w(t, x) := E[\rho](t, x) := E_1[\rho](t, x) + E_2[\rho](t, x)$. We shall need the following estimates, all of which are available in various spots in the literature. For the convenience of the reader, we sketch proofs of these estimates in the appendix. All norms for w and ρ are with respect to the x-variable.

A. Uniform estimate on w:

$$\|w\|_\infty \leqslant C(N, \gamma, \varepsilon)[\|\rho\|_\infty^{(N-1)/N} \|\rho\|_1^{1/N} + \|\nabla\rho\|_\infty^{(N-1)/N} \|\nabla\rho\|_1^{1/N}] \tag{2.1}$$

B. Uniform estimate on ∇w:

For $0 < \alpha \leqslant 1$, let $|g|_{\alpha, N} = \sup_{x, h}(1/|h|^\alpha)\, |g(x+h) - g(x)|$. Then

$$\begin{aligned} \|Dw\|_\infty \leqslant C(N, \gamma, \varepsilon, \alpha)([\|\rho\|_1 + \|\nabla\rho\|_1 + \|\rho\|_\infty \{1 + \ln(1 + \|\nabla\rho\|_\infty)\}] \\ + [\|\nabla\rho\|_\infty + |\nabla\rho|_{\alpha, N}]); \end{aligned} \tag{2.2}$$

C. Hölder estimates:

If $p \in (N, \infty)$ is such that $\alpha = 1 - (N/p)$, then

$$|w|_{\alpha, N} \leqslant C(N, \gamma, \varepsilon, \alpha)[\|\rho\|_p + \|\nabla\rho\|_p] \tag{2.3}$$

and

$$|Dw|_{\alpha, N} \leqslant C(N, \gamma, \alpha)\, \|\nabla\rho\|_p + C(N, \varepsilon, \alpha)\, |\nabla\rho|_{\alpha, N} \tag{2.4}$$

In view of these estimates, we will need to control the following norms and seminorms of $\rho = \int f\, dv$:

(A) $\|\rho\|_1$, (B) $\|\nabla\rho\|_1$, (C) $\|\rho\|_\infty$,

(D) $\|\nabla\rho\|_\infty$ and (E) $|\nabla\rho|_{\alpha, N}$

Recall that all L^p-norms of ρ and $\nabla\rho$ can be bounded in terms of the L^∞-norm and the L^1-norm, by straightforward interpolation.

The following interpolation estimates, whose proofs are also sketched in the appendix, give bounds on the norms from (A)–(E) in terms of the function f. The symbol ω_N denotes the surface measure of the N-dimensional unit sphere. For (C) and (D),

$$\forall \eta > N, \qquad \|\rho\|_\infty \leqslant \eta \frac{\omega_N}{N(\eta - N)} \|f\|_\infty^{1-(N/\eta)} \|(1 + |v|^2)^{\eta/2} f\|_\infty^{N/\eta} \tag{2.5}$$

(the norms on the right are on $\mathbb{R}^{2N}$);

$$\forall \eta > N, \qquad \|\nabla\rho\|_\infty \leqslant \eta \frac{\omega_N}{N(\eta - N)} \|\nabla f\|_\infty^{1-(N/\eta)} \|(1 + |v|^2)^{\eta/2}\, \nabla f\|_\infty^{N/\eta} \tag{2.6}$$

For (E),

$$|\nabla\rho|_{\alpha,N} \leqslant \eta\, \frac{\omega_N}{N(\eta-N)}\, |\nabla f|_{\alpha,2N}^{1-(N/\eta)}\, |(1+|v|^2)^{\eta/2}\, \nabla f|_{\alpha,2N}^{N/\eta} \tag{2.7}$$

3. ESTIMATING SOLUTIONS OF MOLLIFIED EQUATIONS

It is well known that Vlasov equations with smoothed potentials admit global classical solutions, and these results readily generalize to the stellar dynamic Manev equation (1.1). Specifically, if we denote by ω_δ C^∞-approximations of unity such that $\omega_\delta * f \to f$ as $\delta \to 0$, let

$$U[\rho](x) = -\gamma\, \frac{1}{|x|^{N-2}} * \rho - \varepsilon\, \frac{1}{|x|^{N-1}} * \rho$$

and

$$\begin{aligned} U^\delta[\rho](x) &= -\gamma\, \frac{1}{|x|^{N-2}} * \omega_\delta * \rho - \varepsilon\, \frac{1}{|x|^{N-1}} * \omega_\delta * \rho \\ E^\delta[\rho](x) &= \nabla U^\delta[\rho](x) \end{aligned} \tag{3.1}$$

The mollified potential U^δ, and hence the mollified field E^δ, are C^∞ for $\rho \in L^\infty \cap L^1$, and δ- dependent bounds on the L^∞-norms of U^δ and E^δ and their derivatives apply. Note that the mollifier in (3.1) can be thought of as acting on ρ or on the potential kernel—in view of the associativity of the convolution, the result is the same.

It follows that the estimates from Section 2 carry over, uniformly with respect to δ, to the mollified case. Our strategy to solve the system (1.1-3) will be based on proving uniform estimates on solutions of the mollified problem; for simplicity, we shall delete the index δ, but it will implicitly be assumed in the rest of this section that $f = f^\delta$ will be a (global and unique) solution of the initial value problem for (1.1-3), with the E_i's replaced by the E_i^δ's.

For a function $f(t, x, v)$ defined on $[0, T] \times \mathbb{R}^N \times \mathbb{R}^N$, let

$$Y_f(t) := \sup_{0 \leqslant s \leqslant t} \max \begin{bmatrix} \|f(s,\cdot)\|_\infty,\ \|f(s,\cdot)\|_1,\ \|Df(s,\cdot)\|_\infty, \\ \|Df(s,\cdot)\|_1,\ \|(1+|v|^2)^{\eta/2} f(s,\cdot)\|_\infty, \\ \|(1+|v|^2)^{\eta/2}\, Df(s,\cdot)\|_\infty,\ |Df(s,\cdot)|_{\alpha,2N}, \\ |(1+|v|^2)^{\eta/2}\, Df(s,\cdot)|_{\alpha,2N} \end{bmatrix} \tag{3.2}$$

i.e., $Y_f(t)$ is simply the largest value any of the relevant (semi-)norms of f has assumed until time t.

Think of f as a solution of the mollified initial value problem, and of $\rho = \rho^\delta$ as the mollified spatial density associated with this f. The function w defined in Section 2 can then be estimated in terms of $Y_f(t)$. Let $Y_f = \sup_{0 \leqslant t \leqslant T} Y_f(t)$. In what follows, $L_i(\lambda)$, $i = 1, 2, 3,\ldots$ will denote positive, non-decreasing differentiable functions of a variable λ, possibly depending on T (but not on δ). The estimates from Section 2 assert that

$$\|w(t, \cdot)\|_\infty \leqslant L_1(Y_f(t)) \tag{3.3}$$

$$\|Dw(t, \cdot)\|_\infty \leqslant L_2(Y_f(t)) \tag{3.4}$$

$$|w(t, \cdot)|_{\alpha, N} \leqslant L_3(Y_f(t)) \tag{3.5}$$

$$|Dw(t, \cdot)|_{\alpha, N} \leqslant L_4(Y_f(t)) \tag{3.6}$$

and $L_i(Y_f(t)) \leqslant L_i(Y_f)$ for $i = 1,\ldots, 4$.

Our first and most important objective is to obtain uniform bounds (with respect to δ) on the function $Y_f(t)$ on some time interval $[0, T]$ which may depend on the initial values.

We begin by studying the characteristic equations associated with the (mollified) equation (1.1). They are

$$\frac{dX}{ds} = V; \qquad \frac{dV}{ds} = w(s, X), \qquad \text{where} \quad (X, V)|_{s=t} = (x, v) \tag{3.7}$$

Here and in the sequel, we abbreviate $Q = (x, v)$. Integral versions of (3.7) are

$$X(s; t, Q) = x - \int_s^t V(\tau; t, Q)\, d\tau; \qquad V(s; t, Q) = v - \int_s^t w(\tau, X(\tau; t, Q))\, d\tau$$

or equivalently,

$$\begin{aligned} X(s; t, Q) &= x - v(t-s) + \int_s^t (r-s)\, w(r, X(r; t, Q))\, dr \\ V(s; t, Q) &= v - \int_s^t w(\tau, X(\tau; t, Q))\, d\tau \end{aligned} \tag{3.8}$$

We shall abbreviate $P(s; t, Q) = (X(s; t, Q), V(s; t, Q))$ for the solutions of (3.8) with $P_0(t, Q) := P(0; t, Q)$. We need expressions and estimates for the partial derivatives of P with respect to t, x, and v. We have

$$D_x X(s;\, t,\, Q) = I + \int_s^t (r-s)\, D_x w(r,\, X(r;\, t,\, Q))\, D_x X(r;\, t,\, Q)\, dr$$

$$D_x V(s;\, t,\, Q) = -\int_s^t D_x w(r,\, X(r;\, t,\, Q))\, D_x X(r;\, t,\, Q)\, dr$$

$$D_v X(s;\, t,\, Q) = (s-t)\, I + \int_s^t (r-s)\, D_x w(r,\, X(r;\, t,\, Q))\, D_v X(r;\, t,\, Q)\, dr$$

$$D_v V(s;\, t,\, Q) = I - \int_s^t D_x w(r,\, X(r;\, t,\, Q))\, D_v X(r;\, t,\, Q)\, dr$$

$$\begin{aligned}\partial_t X(s;\, t,\, Q) = {} & -v + (t-s)\, w(t,\, x) \\ & + \int_s^t (r-s)\, \nabla_x w(r,\, X(r;\, t,\, Q))\, \partial_t X(r;\, t,\, Q)\, dr\end{aligned}$$

and

$$\partial_t V(s;\, t,\, Q) = -w(t,\, x) - \int_s^t D_x w(r,\, X(r;\, t,\, Q))\, \partial_t X(r;\, t,\, Q)\, dr$$

In differential form,

$$\frac{d}{ds} D_x P(s;\, t,\, Q) = A(s,\, X(s;\, t,\, Q))\, D_x P(s;\, t,\, Q)$$

$$\frac{d}{ds} D_v P(s;\, t,\, Q) = A(s,\, X(s;\, t,\, Q))\, D_v P(s;\, t,\, Q)$$

where

$$A(s,\, X(s;\, t,\, Q)) = \begin{bmatrix} 0 & I \\ Dw(s,\, X(s;\, t,\, Q)) & 0 \end{bmatrix} \tag{3.9}$$

and

$$D_x P(t;\, t,\, Q) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_v P(t;\, t,\, Q) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

The above representations lead to the following.

Lemma 3.1. If $w(x,t)$ is sufficiently smooth, then

$$[\partial_t + v \cdot D_x + w(t, x) \cdot D_v]\, P(s; t, Q) = 0 \tag{3.10}$$

Proof. The above representations for V and its derivatives readily imply that

$$\begin{aligned}
&[\partial_t + v \cdot D_x + w(t, x) \cdot D_v]\, V(s; t, Q)\\
&\qquad = -\int_s^t \nabla_x w(r, X(r; t, Q))[\partial_t + v \cdot D_x + w(t, x) \cdot D_v]\, X(r; t, Q)\, dr
\end{aligned}$$

A similar conclusion holds for $X(s; t, Q)$. (3.10) follows from an application of Gronwall's inequality. ∎

Proposition 3.2. Let $f = f^\delta$ be the solution of the mollified initial value problem, and let $w = w[\rho^\delta]$ be as in Section 2. Then there exists a unique solution $P(s; t, Q)$ of the characteristic equations (3.7). This solution is continuously differentiable with respect to all the variables. For any fixed t and s, $X(s; t, Q)$ and $V(s; t, Q)$ constitute a one-to-one and measure preserving map of $\mathbb{R}^{2N}$ into itself, with Jacobian 1. Furthermore, we have

$$\begin{aligned}
&P(t; t, \cdot) = \mathrm{Id},\\
&P(t; s, \cdot) \text{ is the inverse transformation of } P(s; t, \cdot),\\
&|X(s; t, Q) - x + v(t-s)| \leqslant L_4(Y_f),\\
&|V(s; t, Q) - v| \leqslant L_1(Y_f),\\
&|\partial_t P(s; t, Q)| \leqslant (1 + |v|^2)^{1/2} L_5(Y_f),\\
&|D_Q P(s; t, Q)| \leqslant L_6(Y_f)
\end{aligned} \tag{3.11}$$

Proof. All of the above assertions are straightforward consequences of Gronwall's inequality in conjunction with the estimates from Section 2. ∎

For later estimates involving Hölder semi-norms we need to control $P(s; t, Q) - P(s; t, Q')$ in terms of $Q - Q'$. This is the purpose of the next lemma.

Lemma 3.3. Suppose that $0 \leqslant s \leqslant t$. Then

$$|P(s; t, Q) - P(s; t, Q')| \leqslant |Q - Q'| \exp\Big((t-s) \sup_{s \leqslant \sigma \leqslant t} \|A(\sigma, \cdot)\|_\infty\Big) \tag{3.12}$$

Corollary 3.4.

$$\frac{|P(s;t,Q)-P(s;t,Q')|^{\alpha}}{|Q-Q'|^{\alpha}} \leqslant \exp(\alpha(t-s)\sup_{s\leqslant\sigma\leqslant t}\|A(\sigma,\cdot)\|_{\infty}) \tag{3.13}$$

In view of (2.2) and (3.9) this means that the left-hand side of (3.13) is controlled in terms of the $\|\cdot\|_{\infty}$, $\|\cdot\|_1$ and $|\cdot|_{\alpha,N}$ (semi-)norms of ρ and $\nabla\rho$; these, in turn, are estimated by (2.5)–(2.7).

Proof (of Lemma 3.3.) For $0\leqslant s\leqslant t$,

$$X(s;t,Q)-X(s;t,Q')=x-x'-\int_s^t (V(\tau;t,Q)-V(\tau;t,Q'))\,d\tau$$

$$V(s;t,Q)-V(s;t,Q')=v-v'-\int_s^t (w(\tau,X(\tau;t,Q))-w(\tau,X(\tau;t,Q')))\,d\tau$$

Hence

$$\begin{aligned}&|P(s;t,Q)-P(s;t,Q')|\\ &\quad\leqslant |Q-Q'|+\int_s^t \sup_{s\leqslant\sigma\leqslant t}\|A(\sigma,\cdot)\|_{\infty}\,|P(\tau;t,Q)-P(\tau;t,Q')|\,d\tau\end{aligned}$$

and the assertion follows by Gronwall's inequality. ▮

We next specify the necessary regularity assumptions for the initial value f_0. Specifically, we assume that there is an $\eta>N$ such that

$$\begin{aligned}&(1+|x|^2)^{(1+\eta)/2}(1+|v|^2)^{(1+\eta)/2}[f_0(x,v)+|Df_0(x,v)|\\ &\quad+|D^2f_0(x,v)|]\in L^{\infty}(\mathbb{R}^{2N})\end{aligned} \tag{3.14}$$

Under these conditions, we have the following.

Proposition 3.5. The classical solution of the mollified initial value problem is given by

$$f(t,Q)=f_0(P_0(t,Q)) \tag{3.15}$$

Furthermore,

$$(1+|v|^2)^{-1/2}[\partial_t f, \nabla_x f, \nabla_v f]$$

remains bounded.

Remark. The assertion of this proposition holds on any time interval, but the bounds may depend on the time T and on the mollification parameter δ. Our principal objective is to show that for T small enough, uniform bounds with respect to δ can be obtained.

Proof. Except for the last assertion, everything stated here is well-known. The boundedness assertions follow from the representation (3.15) and the estimates in (3.11). ∎

The next theorem is the most important result in this paper inasmuch as it supplies the relevant tools for a proof of local existence. $C(f_0)$ denotes constants which depend only on f_0; they can be different in different formulas. We denote by D the gradient ∇_P. Ω_T is a shorthand for $[0, T] \times \mathbb{R}^{2N}$, and $B^{\alpha, 1+\alpha}(\Omega_T)$ is the space of functions which are α-Hölder continuous with respect to time, differentiable with respect to $Q = (x, v)$, and whose gradient with respect to Q is Hölder continuous with exponent α.

Theorem 3.6. For f as in (3.15), we have the following identities and estimates:

$$\|f(t, \cdot)\|_\infty = \|f_0\|_\infty, \qquad \|f(t, \cdot)\|_1 = \|f_0\|_1 \tag{3.16}$$

$$|f(t, Q)| \leqslant C(f_0)(1 + |v|^2)^{-\eta/2} L_7(Y_f) \tag{3.17}$$

$$\|f\|_{B^{1, 1+\alpha}(\Omega_T)} \leqslant C(f_0)\, L_8(Y_f) \tag{3.18}$$

$$|Df(t, Q)| \leqslant C(f_0)(1 + |v|^2)^{-\eta/2} L_9(Y_f) \tag{3.19}$$

Df is also Hölder continuous with exponent α with respect to t, and the following stronger version of (3.19) holds:

$$|Df(t, Q)| \leqslant C(f_0)(1 + |v|^2)^{-\eta/2} (1 + (x - vt)^2)^{-\eta/2} L_9(Y_f) \tag{3.19a}$$

Furthermore, we have that

$$\|Df(t, \cdot)\|_1 \leqslant C(f_0)\, L_{10}(Y_f) \tag{3.20}$$

$$|Df(t, \cdot)|_{\alpha, 2N} \leqslant C(f_0)\, L_{11}(Y_f) \tag{3.21}$$

and

$$|(1 + |v|^2)^{\eta/2} Df(t, \cdot)|_{\alpha, 2N} \leqslant C(f_0)\, L_{12}(Y_f) \tag{3.22}$$

Proof. The first assertion in (3.16) is immediate from (3.15). The second assertion is a consequence of Proposition 3.2.

In preparation for (3.17) and (3.19), we next work on velocity decay estimates for f and Df. Let $\chi(t, Q) := (1+|v|^2)^{\eta/2} f(t, Q)$. χ satisfies the equation

$$[\partial_t + v\cdot\nabla_x + w(t,x)\cdot\nabla_v]\,\chi = \eta w\cdot v(1+|v|^2)^{-1}\chi$$

which implies an estimate

$$\sup_{0\leqslant s\leqslant t}\|\chi(s,\cdot)\|_\infty \leqslant \|\chi(0,\cdot)\|_\infty + \eta\int_0^t \sup_{0\leqslant\tau\leqslant s}\|w(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|\chi(\tau,\cdot)\|_\infty\, ds$$

By Gronwall's inequality,

$$\sup_{0\leqslant s\leqslant t}\|\chi(s,\cdot)\|_\infty \leqslant C(f_0)\, L_{13}(Y_f(t))$$

which entails (3.17). Turning to the derivative, we define

$$\xi(t, Q) := (1+|v|^2)^{\eta/2}\, Df(t, Q) \tag{3.23}$$

Roughly speaking, the quantity ξ measures the influence of large velocities on the non-temporal derivatives of f. A formal, straightforward derivation yields the following inhomogeneous Vlasov type equation, which is solved in the mild sense by ξ:

$$[\partial_t + v\cdot\nabla_x + w(t,x)\cdot\nabla_v]\,\xi = -A^t\xi + \eta w\cdot v(1+|v|^2)^{-1}\xi \tag{3.24}$$

where A is the matrix given before Lemma 3.1 (in 3.9); the superscript t denotes transposition.

We integrate (3.24) by the method of characteristics, with the initial value $\xi(0,\cdot) = (1+|v|^2)^{\eta/2}\, Df_0$. We find

$$\begin{aligned}\xi(s, P(s; t, Q)) = {} & (1+|V_0(t,Q)|^2)^{\eta/2}\, Df_0(P_0(t,Q)) \\ & + \int_0^s [\eta(1+|V(r;t,Q)|^2)^{-1}\, V(r;t,Q)\, w(r, X(r;t,Q)) \\ & - A^t(r, X(r;t,Q))]\, \xi(r, P(r;t,Q))\, dr\end{aligned} \tag{3.25}$$

If $\xi(s, P(s; t, Q))$ is sufficiently smooth, the total s-derivative (Lagrangian derivative) is equal to the right hand side of (3.24), evaluated at $(s, P(s; t, Q))$. An expression for $\xi(t, Q)$ in terms of $\xi(r, P(r; t, Q))$ is obtained by setting $s = t$ in (3.25):

$$\xi(t, Q) = (1+|V_0(t,Q)|^2)^{\eta/2}\, Df_0(P_0(t,Q)) + \int_0^t [\,\cdots\,]\, \xi(r, P(r;t,Q))\, dr \tag{3.26}$$

Equation (3.26) is a weak formulation of (3.24) inasmuch as solutions of (3.24) will satisfy (3.26); conversely, by Lemma 3.1, sufficiently smooth solutions of (3.26) will solve (3.24).

The representations (3.25) and (3.26) of ξ allows us to obtain uniform estimates on ξ by a linear Gronwall argument:

$$\begin{aligned}\sup_{0\leqslant\tau\leqslant s} \|\xi(\tau,\cdot)\|_\infty \leqslant{}& \|(1+|v|^2)^{\eta/2}\, Df_0\|_\infty \\ &+\int_0^s \Big[\eta \sup_{0\leqslant\tau\leqslant s'} \|w(\tau,\cdot)\|_\infty + \sup_{0\leqslant\tau\leqslant s'} \|A'(\tau,\cdot)\|_\infty\Big] \\ &\times \sup_{0\leqslant\tau\leqslant s'} \|\xi(\tau,\cdot)\|_\infty\, ds'\end{aligned}$$

We conclude

$$\sup_{0\leqslant s\leqslant t} \|\xi(s,\cdot)\|_\infty \leqslant C(f_0)\, L_{14}(Y_f(t)) \tag{3.27}$$

proving (3.19).

Proof of (3.20). Note that the differential equation satisfied by $Df(t,Q)$ follows from (3.24) by setting $\eta=0$. Just as for $\xi(t,Q)$, we need information about $Df(s, P(s;t,Q))$, which is equal to $Df(t,Q)$ for $s=t$. For $s\leqslant t$, $Df(s,P(s;t,Q))$ satisfies

$$[\partial_t + v\cdot\nabla_x + w(t,x)\cdot\nabla_v]\, Df(t,Q) = -A'(t,x)\, Df(t,Q) \tag{3.28}$$

with $Df|_{t=0} = Df_0$.

The vector field $Df(s,P(s;t,Q))$ can therefore be written as a solution of the integral equation

$$Df(s,P(s;t,Q)) = Df_0(P_0(t,Q)) - \int_0^s A'(\tau, X(\tau;t,Q))\, Df(\tau, P(\tau;t,Q))\, d\tau \tag{3.29}$$

and by Gronwall's inequality, with t, Q considered as parameters,

$$|Df(s,P(s;t,Q))| \leqslant |Df_0(P_0(t,Q))| \exp\left\{\int_0^s \|A'(\tau,\cdot)\|_\infty\, d\tau\right\}$$

By setting $s=t$, we find

$$|Df(t,Q)| \leqslant |Df_0(P_0(t,Q))| \exp\left\{\int_0^t \sup_{0\leqslant s\leqslant\tau} \|A'(s,\cdot)\|_\infty\, d\tau\right\} \tag{3.30}$$

In the sequel, we use the elementary inequality

$$(1+|x|^2)^{1/2}\,(1+|x'|^2)^{-1/2} \leqslant 1+|x-x'| \tag{3.31}$$

which holds for $x, x' \in \mathbb{R}^N$. By (3.31), (3.30) and (3.15), it follows that

$$\begin{aligned}|Df(t, Q)| &\leqslant C(f_0)\, L_{15}(Y_f(t))(1+|X_0(t, Q)|^2)^{-\eta/2}\,(1+|V_0(t, Q)|^2)^{-\eta/2}\\ &\leqslant C(f_0)\, L_{16}(Y_f(t))(1+|x-vt|^2)^{-\eta/2}\,(1+|v|^2)^{-\eta/2}\\ &\quad\times\left[1+\int_0^t r\,\|w(r,\cdot)\|_\infty\,dr\right]^\eta\left[1+\int_0^t \|w(r,\cdot)\|_\infty\,dr\right]^\eta\end{aligned} \tag{3.32}$$

where in the last estimate we took advantage of (3.8) and (3.31). The estimate (3.20) is now immediate from (3.32), and we have also obtained (3.19a).

Proofs of (3.18), (3.21) and (3.22). To this end, we need information about the Hölder continuity of $Df(s, P(s; t, Q))$ and $\xi(s, P(s; t, Q))$ with respect to s, t and $Q=(x, v)$. Let $t'>t$. Then

$$(Df)(t', Q) = Df_0(P_0(t', Q)) + \int_0^{t'} A^t(r, X(r; t', Q))\, Df(r, P(r; t', Q))\, dr$$

and similarly for $(Df)(t, Q)$. We also consider

$$\begin{aligned}&Df(s, P(s; t', Q)) - Df(s, P(s; t, Q))\\ &\quad= Df_0(P_0(t', Q)) - Df_0(P_0(t, Q))\\ &\qquad+\int_0^s [A(r, X(r; t', Q)) - A(r, X(r, t, Q))]^t\, Df(r, P(r; t', Q))\, dr\\ &\qquad+\int_0^s A^t(r, X(r; t, Q))[Df(r, P(r; t', Q)) - Df(r, P(r; t, Q))]\, dr\end{aligned} \tag{3.33}$$

We investigate the behavior of the expression on the left-hand side of (3.33); an estimate on $(Df)(t', Q) - (Df)(t, Q)$ will follow, due to the fact that

$$\begin{aligned}Df(t', Q) - Df(t, Q) &= \int_t^{t'} A^t(r, X(r; t', Q))\, Df(r, P(r; t', Q))\, dr\\ &\quad+ [Df(s, P(s; t', Q)) - Df(s, P(s; t, Q))]|_{s=t}\end{aligned} \tag{3.34}$$

By Gronwall's inequality, we conclude from (3.33) that

$$
\begin{aligned}
&|Df(s, P(s; t', Q)) - Df(s, P(s; t, Q))| \\
&\quad \leqslant |Df_0(P_0(t', Q)) - Df_0(P_0(t, Q))| \exp\left\{\int_0^s \|A'(\tau, \cdot)\|_\infty \, d\tau\right\} \\
&\quad\quad + \int_0^s \|[A(s', X(s'; t', Q)) - A(s', X(s', t, Q))]'\|_\infty \\
&\quad\quad \times |Df(s', P(s', t', Q))| \exp\left\{\int_{s'}^s \|A'(\tau, \cdot)\|_\infty \, d\tau\right\} ds' \\
&\quad =: I + II
\end{aligned}
\tag{3.35}
$$

For I, we write

$$
\begin{aligned}
I &\leqslant \exp\left\{\int_0^s \|A'(\tau, \cdot)\|_\infty \, d\tau\right\} \int_0^1 D^2 f_0(P_0(t, Q) + \theta(P_0(t', Q) - P_0(t, Q))) \, d\theta \\
&\quad \times |P_0(t', Q) - P_0(t, Q)| \\
&\leqslant C(f_0)\, L_{17}(Y_f(t))(1 + |v|^2)^{1/2} \, |t - t'| \\
&\quad \times \int_0^1 [1 + |V_0(t, Q)(1 - \theta) + \theta V(t', Q)|^2]^{-\eta/2} \, d\theta \\
&\leqslant C(f_0)\, L_{18}(Y_f(t))(1 + |v|^2)^{(1-\eta)/2} \, |t - t'| \\
&\leqslant C(f_0)\, L_{18}(Y_f(t)) |t - t'|
\end{aligned}
$$

so this term is actually Lipschitz continuous with respect to t.

We use the properties of the Manev forces and the Hörmander lemma (see the appendix) to estimate II. We can estimate

$$
\begin{aligned}
II &= \int_0^s \|[A(s', X(s'; t', Q)) - A(s', X(s', t, Q))]'\|_\infty \, |Df(s', P(s', t', Q))| \\
&\quad \times \exp\left\{\int_{s'}^s \|A'(\tau, \cdot)\|_\infty \, d\tau\right\} ds' \\
&\leqslant L_{19}(Y_f(t)) \int_0^s |X(s'; t', Q) - X(s'; t, Q)|^\alpha \, |Df(s', P(s', t', Q))| \, ds' \\
&\leqslant L_{20}(Y_f(t)) |t - t'|^\alpha \, (1 + |v|^2)^{(\alpha - \eta)/2} \, |Df_0(P_0(t', Q))| \\
&\leqslant C(f_0)\, L_{20}(Y_f(t)) |t - t'|^\alpha
\end{aligned}
\tag{3.36}
$$

(Eq. (3.11) has been used in the second to last line). It follows that

$$|Df(t, Q) - Df(t', Q)| \leqslant C(f_0)\, L_{21}(Y_f(t))|t - t'|^\alpha \tag{3.37}$$

bounding the Hölder semi-norm of Df with respect to time.

We next show Lipschitz continuity of f with respect to t; together with the subsequent proof of (3.21), this will prove (3.18).

$$\begin{aligned}
&|f(t', Q) - f(t, Q)| \\
&\quad = |f_0(P_0(t', Q)) - f_0(P_0(t, Q))| \\
&\quad \leqslant \int_0^1 |Df_0(P_0(t, Q) + \theta(P_0(t', Q) - P_0(t, Q))|\, d\theta\, |P_0(t', Q) - P_0(t, Q)| \\
&\quad \leqslant C(f_0)\, L_5(Y_f(t))|t - t'|(1 + |v|^2)^{1/2} \\
&\qquad \times \sup_\theta \left[1 + |P_0(t, Q) + \theta(P_0(t', Q) - P_0(t, Q))|\right]^{-\eta/2} \\
&\quad \leqslant C(f_0)\, L_{22}(Y_f(t))(1 + |v|^2)^{-(\eta+1)/2}\, |t - t'|
\end{aligned}$$

To prove Hölder continuity of $Df(t, Q)$ with respect to x and v, we set $Q = (x, v)$, $Q' = (x', v')$, and write

$$\begin{aligned}
&Df(s, P(s; t, Q')) - Df(s, P(s; t, Q)) \\
&\quad = Df_0(P_0(t, Q')) - Df_0(P_0(t, Q)) \\
&\qquad + \int_0^s [A(r, X(r; t, Q')) - A(r, X(r; t, Q))]^t\, Df(r, P(r; t, Q'))\, dr \\
&\qquad + \int_0^s A(r, X(r; t, Q))^t\, [Df(r, P(r; t, Q')) - Df(r, P(r; t, Q))]\, dr
\end{aligned} \tag{3.38}$$

Dividing (3.38) by $|Q - Q'|^\alpha$, we find

$$\begin{aligned}
|Df(s, P(s; t, \cdot))|_{\alpha, 2N} &\leqslant |Df_0|_{\alpha, 2N} \sup_{0 \leqslant s \leqslant t} \frac{|P_0(s, Q) - P_0(s, Q')|^\alpha}{|Q - Q'|^\alpha} \\
&\quad + \int_0^s \sup_{0 \leqslant \sigma \leqslant r} |A(\sigma, \cdot)|_{\alpha, N} \sup_{0 \leqslant \sigma \leqslant r} \|Df(\sigma, \cdot)\|_\infty
\end{aligned}$$

$$\times \sup_{0 \leqslant \sigma \leqslant r} \frac{|X(\sigma; t, Q) - X(\sigma; t, Q')|^\alpha}{|Q - Q'|^\alpha} dr$$

$$+ \int_0^s |Df(r; P(r; t, \cdot))|_{\alpha, 2N} \sup_{0 \leqslant \sigma \leqslant r} (\|A(\sigma, \cdot)\|_\infty)$$

$$\times \sup_{0 \leqslant \sigma \leqslant r} \frac{|P(\sigma; t, Q) - P(\sigma; t, Q')|^\alpha}{|Q - Q'|^\alpha} dr$$

By Corollary 3.4 we have

$$|Df(s, P(s; t, \cdot))|_{\alpha, 2N}$$

$$\leqslant |Df_0|_{\alpha, 2N} \exp(\alpha t \sup_{s \leqslant \sigma \leqslant t} \|A(\sigma, \cdot)\|_\infty)$$

$$+ \int_0^s \sup_{0 \leqslant \sigma \leqslant r} |A(\sigma, \cdot)|_{\alpha, N} \exp(\alpha t \sup_{s \leqslant \sigma \leqslant t} \|A(\sigma, \cdot)\|_\infty) \sup_{0 \leqslant \sigma \leqslant r} \|Df(\sigma, \cdot)\|_\infty dr$$

$$+ \int_0^s \sup_{0 \leqslant \sigma \leqslant r} \|A(\sigma, \cdot)\|_\infty \exp(\alpha t \sup_{s \leqslant \sigma \leqslant r} \|A(\sigma, \cdot)\|_\infty) |Df(r, P(r; t, \cdot)|_{\alpha, 2N} dr$$

and by Gronwall's inequality $\sup_{0 \leqslant s \leqslant t} |Df(s, P(s; t, \cdot))|_{\alpha, 2N}$ is bounded in terms of

$$|Df_0|_{\alpha, 2N}, \quad \sup_{0 \leqslant s \leqslant t} \|Dw(s, \cdot)\|_\infty, \quad \sup_{0 \leqslant s \leqslant t} \|Df(s, \cdot)\|_\infty$$

and $\sup_{0 \leqslant s \leqslant t} |Dw(s, \cdot)|_{\alpha, N}$. As $Df(s, P(s; t, Q))|_{s=t} = Df(t, Q)$, we get in particular control of the Hölder semi-norm of $Df(t, \cdot)$. This proves (3.21).

We finally investigate the Hölder continuity of $\xi(t, x, v)$, needed for (3.22). Recall that

$$\xi(t, Q) = \xi_0(P_0(t, Q)) + \int_0^t R(r, P(r; t, Q)) \, \xi(r, P(r; t, Q)) \, dr \tag{3.39}$$

with

$$R(\cdots) = \eta[1 + |V(r; t, Q)|^2]^{-1} V(\cdots) \cdot w(r, X(r; t, Q)) + A^t(\cdots) \tag{3.40}$$

It is necessary to study $\xi(s, P(s; t, Q))$, which satisfies (3.25), i.e.,

$$\xi(s, P(s; t, Q)) = \xi_0(P_0(t, Q)) + \int_0^s R(r, P(r; t, Q))\, \xi(r, P(r; t, Q))\, dr$$

From the assumptions on the initial value, and the estimates (3.11), it follows that

$$\begin{aligned}
&|\xi(s, P(s; t, Q))| \\
&\quad \leqslant |\xi_0(P_0(t, Q))| e^{\int_0^t \|R(\tau, \cdot)\|_\infty\, d\tau} \\
&\quad \leqslant C(f_0)\, L_{23}(Y_f(t))(1 + |V_0(t, Q)|^2)^{\eta/2}\, (1 + |V_0(t, Q)|^2)^{(-\eta-1)/2} \\
&\qquad \times (1 + |X_0(t, Q)|^2)^{-\eta/2} \\
&\quad \leqslant C(f_0)\, L_{24}(Y_f(t))(1 + |v|^2)^{-1/2}\, (1 + |x - vt|^2)^{-\eta/2} \qquad (3.41)
\end{aligned}$$

Hölder continuity of $\xi(s, P(s; t, x, v))$ with respect to t also applies and is obtained as follows. The usual Gronwall analysis yields

$$\begin{aligned}
&|\xi(s, P(s; t', Q)) - \xi(s, P(s; t, Q))| \\
&\leqslant |\xi_0(P_0(t', Q)) - \xi_0(P_0(t, Q))| \exp\left[\int_0^s \|R(\tau, \cdot)\|_\infty\, d\tau\right] \\
&\quad + \int_0^s [R(\tau, P(\tau; t', Q)) - R(\tau, P(\tau; t, Q))]\, e^{\int_\tau^s \|R(\sigma, \cdot)\|_\infty\, d\sigma}\, |\xi(\tau, P(\tau; t', Q))|\, d\tau
\end{aligned} \qquad (3.42)$$

The properties of the Manev force term $A'(\tau, X(\tau; t, Q))$ in the integrand of the second term on the right hand side lead, as in the estimate (3.36), to an upper bound

$$C(f_0)\, L_{25}(Y_f(t))|t - t'|^\alpha\, (1 + |v|^2)^{-1/2}\, (1 + |x - vt|^2)^{-\eta/2}$$

by (3.41). The first term on the right of (3.42) is estimated by

$$C(f_0)\, L_{26}(Y_f(t))|\xi_0(P_0(t', Q)) - \xi_0(P_0(t, Q))| \leqslant C(f_0)\, L_{26}(Y_f(t))|t - t'|^\alpha$$

by an argument similar to the proof of Lemma 3.3 and (3.14). Hölder continuity of $\xi(s, P(s; t, Q))$ with respect to t follows.

The final step before us is to prove Hölder continuity of $\xi(s, P(s; t, Q))$ with respect to $Q = (x, v)$. This is done in a way similar to the proof of Hölder continuity of $Df(s, P(s; t, Q))$: We write

$$
\begin{aligned}
&\xi(s, P(s; t, Q)) - \xi(s, P(s; t, Q')) \\
&\quad = \xi_0(P_0(t, Q)) - \xi_0(P_0(t, Q')) \\
&\qquad + \int_0^s (R(r, P(r; t, Q)) - R(r, P(r; t, Q')))\, \xi(r, P(r; t, Q))\, dr \\
&\qquad + \int_0^s R(r, P(r; t, Q'))(\xi(r, P(r; t, Q)) - \xi(r, P(r; t, Q')))\, dr
\end{aligned}
$$

Hölder continuity now follows as for $Df(s, P(s; t, Q))$: By the properties of the Manev force terms, and by Corollary 3.4,

$$
\|R(r, P(r; t, Q)) - R(r, P(r; t, Q'))\| \leqslant CL_{27}(Y_f(t))|Q - Q'|^\alpha
$$

(recall $R(\cdots) = \eta(1 + |V(r; t, Q)|^2)^{-1}\, V(\cdots) \cdot w(\cdots) + A'(\cdots)$), and

$$
\begin{aligned}
&|\xi_0(P_0(t, Q)) - \xi_0(P_0(t, Q'))| \\
&\quad \leqslant |(1 + |V(0, Q)|^2)^{\eta/2}\, Df_0(P_0(t, Q)) - (1 + |V(0, Q')|^2)^{\eta/2}\, Df_0(P_0(t, Q'))| \\
&\quad \leqslant |D[(1 + V^2(0, Z))^{\eta/2}\, Df_0(P_0(t, Z))]| \cdot \|Q - Q'\|
\end{aligned}
$$

where we have used the mean value theorem and Z is a point on the line connecting Q and Q'. In view of (3.12) and (3.10), this last expression is bounded by $L_{27}(Y_f(t))\|Q - Q'\|$, hence even Lipschitz-continuity with respect to Q holds for this term. The assertion (3.22) follows by collecting terms. ∎

The function $Y_f(t)$ defined by (3.2) satisfies a Gronwall inequality, obtained by suitably combining those for estimating the right-hand sides of (2.5)–(2.7). The proof of Theorem (3.6) indicates how such inequalities are obtained from equations (3.24), (3.25), (3.28)–(3.29) and (3.38)–(3.39).

By collecting all these estimates together, we arrive at

Theorem 3.7. There is a $C > 0$ and a smooth, monotone increasing function G: $[0, \infty) \to [0, \infty)$, independent of δ, such that

$$
Y_f(t) \leqslant CY_f(0) + \int_0^t G(Y_f(\tau))\, d\tau \tag{3.43}
$$

Hence, if $\zeta(t)$ is the unique local solution of the initial value problem

$$
\zeta' = G(\zeta), \qquad \zeta(0) = CY_f(0)
$$

it follows that

$$Y_f(t) \leqslant \zeta(t) \tag{3.44}$$

The estimate (3.44) implies local-in time uniform control on all the relevant (semi-)norms of the mollified problems (recall that $f = f^\delta$, where δ is the mollification parameter). We use these estimates in the next section to prove the existence of a local solution.

4. EXISTENCE OF A LOCAL SOLUTION

Choose a $T > 0$ such that $0 < T < T_\infty$, where T_∞ is such that $\lim_{t \to T_\infty} \zeta(t) = \infty$. We set $\lambda_0 = \zeta(T)$. Observe that λ_0 is uniquely determined by η and f_0, i.e., the data and its decay properties. We now consider the family $\{f^\delta(t, Q): (t, Q) \in \Omega_T\}$. The following lemma will be essential in the subsequent analysis.

Lemma 4.1. Let g^δ be a family of functions which is equicontinuous and uniformly bounded on Ω_T. Moreover, suppose that for every $\mu > 0$, there is an $R > 0$ such that for all $\delta > 0$ and for $(t, Q) \in \Omega_T$ with $|Q| \geqslant R$,

$$|g^\delta(Q, t)| \leqslant \mu$$

Then there is a subsequence of g^δ which converges uniformly on Ω_T.

Proof. This follows from the classical Arzela–Ascoli Theorem and a standard diagonalization argument.

Theorem 4.2. (Convergence of f^δ.) The family $\{f^\delta\}$ of solutions of mollified Vlasov–Manev equations with the initial value f_0 satisfies the conditions of Lemma 4.1 on Ω_T.

Proof. From Theorem 3.6, we see that

$$|f^\delta(t, Q)| \leqslant \|f_0\|_\infty, \qquad |Df^\delta(t, Q)| \leqslant C(f_0)\, L_9(\zeta(t))(1 + |v|^2)^{-\eta/2}$$

These estimates show uniform boundedness and equicontinuity with respect to the spatial and momentum variables. So all we have to show is equicontinuity with respect to t, and that for every $\mu > 0$, $\exists R > 0$ such that uniformly in δ

$$|f^\delta(t, Q)| \leqslant \mu \qquad \text{if} \quad |Q| \geqslant R \tag{4.1}$$

We first show (4.1). Recall that $f^\delta(t, Q) = f_0(P_0^\delta(t, Q))$ (we have added the index δ to the flow to emphasize the δ-dependence!). Hence

$$\begin{aligned} |f^\delta(t, Q)| &\leqslant C(f_0)(1 + |V_0^\delta(t, Q)|^2)^{-\eta/2} (1 + |X_0^\delta(t, Q)|^2)^{-\eta/2} \\ &\leqslant C(f_0, T)(1 + |v|^2)^{-\eta/2} (1 + |x - tv|^2)^{-\eta/2} \end{aligned}$$

where we have used that

$$X_0^\delta(t, Q) = x - vt + \int_0^t r w^\delta(r, X^\delta(r; t, Q))\, dr$$

$$V_0^\delta(t, Q) = v - \int_0^t w^\delta(r, X^\delta(r; t, Q))\, dr$$

(see (3.8)) and the uniform control on the integrals in terms of various norms of f^δ, i.e., in terms of $\zeta(T)$. Equation (4.1) follows immediately.

The uniform estimates on the force fields, as given in Section 2, and the uniform control on the derivatives Df^δ in terms of $\zeta(t)$, imply equicontinuity in t (it simply follows from the mollified equations that $\{(\partial/\partial t) f^\delta\}$ is uniformly bounded on Ω_T). This completes the proof. ∎

It follows that we can extract a subsequence of $\{f^\delta\}_{\delta > 0}$, again denoted by $\{f^\delta\}$, which will converge uniformly to a limit f, clearly the candidate for a solution. To show that f solves the Vlasov–Manev system (1.1)–(1.3), we have to obtain uniform bounds and decay estimate on Df^δ.

We already know from Theorem 3.6 that

$$|Df^\delta(t, Q)| \leqslant C(f_0)\, L(\zeta(T))(1 + |v|^2)^{-\eta/2}$$

which gives uniform boundedness (and decay in v) for Df^δ. In fact, we also have uniform Hölder continuity of the functions Df^δ, with respect to t and Q, by (3.21) and (3.37). The decay estimates follow from (3.32). It follows that we can extract a uniformly convergent subsequence from Df^δ, with a continuous limit g on Ω_T, and as $f^\delta \to f$, we conclude $g = Df$, and Df is continuous.

We next address convergence of the fields, $E^\delta \to E$. Note that the estimates of $E[\rho^\delta](t, x) - E[\rho](t, x)$ which arise from (2.1) do not require Hölder continuity of ρ^δ or ρ! We must, however, examine the convergences

(i) $f^\delta(t, \cdot) \to f(t, \cdot)$ in $L^1(\mathbb{R}^{2N})$ $\forall t \in [0, T)$,

(ii) $Df^\delta(t, \cdot) \to Df(t, \cdot)$ in $L^1(\mathbb{R}^{2N})$ $\forall t \in [0, T)$,

(iii) $(1+|v|^2)^{\eta/2} f^\delta(t,\cdot) \to (1+|v|^2)^{\eta/2} f(t,\cdot)$ uniformly on Ω_T,

(iv) $(1+|v|^2)^{\eta/2} Df^\delta(t,\cdot) \to (1+|v|^2)^{\eta/2} Df(t,\cdot)$ uniformly on Ω_T,

which will entail the desired convergence $E^\delta \to E$ in L^∞.

The estimates (i) and (ii) are consequences of the Lebesgue dominated convergence theorem and (3.32). To show that $E(t, x)$ is continuous in t uniformly with respect to x, we need to show in addition that $\|f^\delta(t,\cdot)\|_1$ and $\|Df^\delta(t,\cdot)\|_1$ are equicontinuous on $[0, T]$. The convergences (i)–(iv) and these equicontinuities are sufficient to show that $E(t, x)$, defined as $E[\rho](t, x)$, is the uniform limit of the approximate fields E^δ in the sense that

$$\lim_{\delta \searrow 0} \sup_{0 \leqslant s \leqslant t} \|E^\delta(s,\cdot) - E(s,\cdot)\|_\infty = 0$$

and that $\|E(t,\cdot)\|_\infty$ is indeed continuous.

Concerning $\|f^\delta(s,\cdot) - f^\delta(t,\cdot)\|_1$, we write

$$\begin{aligned}
&|f^\delta(s,Q) - f^\delta(t,Q)| \\
&\quad = |f_0(P_0^\delta(s,Q)) - f_0(P_0^\delta(t,Q))| \\
&\quad \leqslant |P_0^\delta(s,Q) - P_0^\delta(t,Q)| \left| \int_0^1 Df_0(P_0^\delta(s,Q) + \theta(P_0^\delta(t,Q) - P_0^\delta(s,Q)))\, d\theta \right|
\end{aligned}$$

and by Theorem 3.6 the last product is

$$\begin{aligned}
&\leqslant C(f_0)\, L_5(\zeta(T))(1+|v|^2)^{1/2}\, |t-s| \\
&\quad \times \int_0^1 (1 + |X_0^\delta(t,Q)(1-\theta) + \theta X_0^\delta(s,Q)|^2)^{-(1+\eta)/2} \\
&\quad \times (1 + |V_0^\delta(t,Q)(1-\theta) + \theta V_0^\delta(s,Q)|^2)^{-(1+\eta)/2}\, d\theta
\end{aligned}$$

By Proposition 3.2,

$$\begin{aligned}
&|X_0^\delta(t,Q)(1-\theta) + \theta X_0^\delta(s,Q)|^2 \\
&\quad = |(x-vt)(1-\theta) + (x-vs)\theta + O(L_4(\zeta))|^2 \\
&\quad = |x - (\theta s + (1-\theta)\, t)\, v + O(L_4(\zeta))|^2
\end{aligned}$$

hence

$$|f^\delta(s, Q) - f^\delta(t, Q)| \leqslant C(f_0)\, L_6(\zeta(T))(1+|v|^2)^{1/2}\, |t-s|\, (1+|v|^2)^{-(1+\eta)/2} \times \int_0^1 (1+|x-(\theta s+(1-\theta)\, t)\, v|^2)^{-(\eta/2)}\, d\theta$$

This estimate implies equicontinuity of $\|f^\delta(s, \cdot)\|_1$ with respect to s on $[0, T]$. In particular, $\|f(s, \cdot) - f(t, \cdot)\|_1 \to 0$ as $t \to s$.

The corresponding result for $\|Df^\delta(s, \cdot)\|_1$ requires some additional analysis, which we provide in several steps.

First, the inequality preceding (3.30) and (3.32) imply

$$|Df^\delta(s, P^\delta(s; t, Q))| \leqslant C(f_0)\, L(\zeta(T))(1+|x-vt|^2)^{-(1+\eta)/2}\, (1+|v|^2)^{-(1+\eta)/2}$$

in view of the assumptions on f_0. So the first term on the right of (3.34), when integrated with respect to Q, yields a term $O(|t-t'|)$, and this is uniform in δ due to the a priori estimates on $\|A^\iota(t, \cdot)\|_\infty$. For the second term in (3.34), focus on (3.35) and its estimate on

$$Df^\delta(s, P^\delta(s; t, Q)) - Df^\delta(s, P^\delta(s; t', Q))$$

The L^1-bound on $|Df_0(P_0^\delta(t, Q)) - Df_0(P_0^\delta(t', Q))|$ is done just as in the earlier treatment of $f_0(P_0^\delta(t, Q)) - f_0(P_0^\delta(t', Q))$. The second integral on the right of (3.35)—when integrated—will yield a term of the type $|t-t'|^\alpha$ by our previous estimates on $Df^\delta(s, P^\delta(s; t, Q))$ (see above) and the estimate used in (3.36), i.e.,

$$|A^\iota(s, X(s; t', Q)) - A^\iota(s, X(s; t, Q))| \leqslant L(\zeta(T))(1+|v|^2)^{\alpha/2}\, |t-t'|^\alpha$$

In conclusion,

$$\|Df^\delta(s, \cdot)\|_1$$

is equicontinuous in s, uniformly in δ; it follows that

$$\|Df(s, \cdot) - Df(t, \cdot)\|_1 \to 0$$

as $s \to t$.

We next consider the families

$$\chi^\delta(t, \cdot) = (1+|v|^2)^{\eta/2}\, f^\delta(t, \cdot)$$

and

$$\xi^\delta(t,\cdot) = (1+|v|^2)^{\eta/2}\, Df^\delta(t,\cdot)$$

$\{\chi^\delta\}$ is uniformly bounded by the estimates preceding (3.23), and uniformly equicontinuous in Q because of the boundedness of ξ^δ. We have to check equicontinuity in t, uniformly in Q. To this end, note that

$$\partial_t \chi^\delta = \eta(1+|v|^2)^{-1}\, v\cdot E[\rho^\delta]\,\chi^\delta - v\cdot\nabla_x\chi^\delta - E[\rho^\delta]\cdot\nabla_v\chi^\delta$$

The first and third terms on the right are uniformly bounded by our previous estimates. As for the term $v\cdot\nabla\chi^\delta$, note that by (3.30)–(3.33) and by (3.14)

$$\begin{aligned}
|Df^\delta(t,Q)| &\leqslant C(f_0)\, L(\zeta(T))(1+|X_0(t,Q)|^2)^{-(1+\eta)/2}\\
&\quad\times(1+|V_0(t,Q)|^2)^{-(1+\eta)/2}\\
&\leqslant C(f_0)\, L(\zeta(T))(1+|x-vt|^2)^{-(1+\eta)/2}\,(1+|v|^2)^{-(1+\eta)/2}
\end{aligned}$$

Hence $|(1+|v|^2)^{(1+\eta)/2}\, Df(t,Q)| \leqslant C(f_0)\, L(\zeta(T))$, $v\cdot\nabla_x\chi^\delta$ is bounded, and uniform boundedness of $\partial_t\chi^\delta$, hence equicontinuity of $\{\chi^\delta\}$ with respect to t, follow. We conclude that there is a subsequence of $\{\chi^\delta\}$ which converges uniformly on Ω_T to its pointwise limit $(1+|v|^2)^{\eta/2}\, f(t,Q)$.

We have succeeded in extracting a subsequence of $\{f^\delta\}$ such that the convergences (i)–(iii) hold. For (iv), we analyse $\{\xi^\delta(t,Q)\colon (t,Q)\in\Omega_T\}$. By (3.27), $\{\xi^\delta\}$ is uniformly bounded on Ω_T, and by the discussion following (3.42), we have equicontinuity. δ-uniform decay with respect to x and v was established in (3.41).

With these observations, a subsequence of $\{\xi^\delta\}$ converges uniformly to its pointwise limit $(1+|v|^2)^{\eta/2}\, Df$ in Ω_T, and we conclude that $E^\delta \to E$ uniformly on $[0,T]\times\mathbb{R}_x^N$.

The limiting function f is a classical solution of the Vlasov–Poisson–Manev system (1.1)–(1.3) for the initial value f_0. We have shown the following.

Theorem 4.3. Suppose that f_0 satisfies the conditions (3.14). Then there is a time interval $[0, T_\infty)$, $T_\infty < \infty$, such that the Vlasov–Poisson–Manev system (1.1)–(1.3) with the initial value f_0 has a classical solution on $[0, T_\infty)$.

Remark. As discussed in part *I* of our work [BDIV], T_∞ is in general finite and will depend on f_0. The above theorem is therefore a local theorem. We also showed in part I that the solution in the "pure" Manev case $\gamma = 0$ will exist globally for certain data if local solutions exist (see Section 3 of [BDIV]).

5. UNIQUENESS

If we require slightly more regularity from the intial value f_0, the solution given by Theorem 4.3 is unique in its class. Specifically, suppose that

$$(1+|x|^2)^{(1+\eta)/2}(1+|v|^2)^{(1+\eta)/2}[|f_0|+|Df_0|+|D^2f_0|+|D^3f_0|] \in L^\infty(\mathbb{R}^{2N})$$

The analysis of the previous two sections shows that a solution will possess the extra regularity in its second derivatives. Suppose that f and $\tilde{f}$ are two solutions associated with the initial value f_0, with I denoting a common interval of existence. The relevant forces are

$$E[\rho](t,x) = -\gamma(N-2)[x\,|x|^{-N} * \rho](t,x) - \varepsilon[|x|^{-N+1} * \rho](t,x) \qquad (5.1)$$

Let $\Delta(t,Q) = \tilde{f}(t,Q) - f(t,Q)$. Δ satisfies

$$(\partial_t + v\cdot\nabla_x + E[\tilde{\rho}](t,x)\cdot\nabla_v)\,\Delta(t,Q) = -w_\Delta(t,x)\,\nabla_v f(t,Q)$$

with

$$\begin{aligned} w_\Delta(t,x) &= E[\tilde{\rho}-\rho](t,x) \\ &= -\gamma(N-2)[x\,|x|^{-N} * (\tilde{\rho}-\rho)](t,x) \\ &\quad - \varepsilon[|x|^{-N+1} * \nabla_x(\tilde{\rho}-\rho)](t,x) \end{aligned} \qquad (5.2)$$

For the derivatives, we have

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x + E[\tilde{\rho}](t,x)\cdot\nabla_v)\,D\Delta(t,Q) \\ &\quad = \tilde{A}(t,x)\,D\Delta(t,Q) - Dw_\Delta(t,x)\cdot\nabla_v f(t,Q) - w_\Delta(t,x)\cdot\nabla Df(t,Q) \end{aligned} \qquad (5.3)$$

and the velocity moments $\chi_\Delta(t,Q) := (1+|v|^2)^{\eta/2}\,\Delta(t,Q)$ and $\xi_\Delta(t,Q) := (1+|v|^2)^{\eta/2}\,D\Delta(t,Q)$ are determined by

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x + E[\tilde{\rho}](t,x)\cdot\nabla_v)\,\chi_\Delta(t,Q) \\ &\quad = \eta(1+|v|^2)^{-1}\,v\cdot E[\tilde{\rho}](t,x)\,\chi_\Delta(t,Q) - w_\Delta(t,x)(1+|v|^2)^{\eta/2}\,\nabla_v f(t,Q) \end{aligned} \qquad (5.4)$$

and

$$(\partial_t + v\cdot\nabla_x + E[\tilde{\rho}](t,x)\cdot\nabla_v)\,\xi_\Delta(t,Q)$$
$$= \eta(1+|v|^2)^{-1}\,v\cdot E[\tilde{\rho}](t,x)\,\xi_\Delta(t,Q) + \tilde{A}(t,x)\,\xi_\Delta(t,Q)$$
$$- Dw_\Delta(t,x)(1+|v|^2)^{\eta/2}\,\nabla_v f(t,Q) - w_\Delta(t,x)(1+|v|^2)^{\eta/2}\,D\nabla_v f(t,Q) \tag{5.5}$$

As before, we must control the growth in time of the quantity

$$\begin{aligned} Y_\Delta(t) := \sup_{0\leqslant s\leqslant t}\ \max[&\|\Delta(s,\cdot)\|_\infty, \|\Delta(s,\cdot)\|_1,\\ &\|D\Delta(s,\cdot)\|_1, \|D\Delta(s,\cdot)\|_\infty,\\ &\|(1+|v|^2)^{\eta/2}\,\Delta(s,\cdot)\|_\infty, \|(1+|v|^2)^{\eta/2}\,D\Delta(s,\cdot)\|_\infty,\\ &|D\Delta(s,\cdot)|_{\alpha,2N}, |(1+|v|^2)^{\eta/2}\,D\Delta(s,\cdot)|_{\alpha,2N}] \end{aligned}$$

The following are the controlling estimates, which follow readily from (5.1)–(5.5) (the one for $|(1+|v|^2)^{\eta/2}\,D\Delta(s,\cdot)|_{\alpha,2N}$ is not displayed, but is similar to the one for $|D\Delta(s,\cdot)|_{\alpha,2N}$):

(i) $$\sup_{0\leqslant s\leqslant t}\|\Delta(s,\cdot)\|_p \leqslant \int_0^t \sup_{0\leqslant\tau\leqslant s}\|w_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|\nabla_v f(\tau,\cdot)\|_p\,ds$$

($p=1$ or $p=\infty$);

(ii) $$\begin{aligned} \sup_{0\leqslant s\leqslant t}\|D\Delta(s,\cdot)\|_p \leqslant \int_0^t [&\sup_{0\leqslant\tau\leqslant s}\|\tilde{A}(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|D\Delta(\tau,\cdot)\|_p\\ &+\sup_{0\leqslant\tau\leqslant s}\|Dw_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|\nabla_v f(\tau,\cdot)\|_p\\ &+\sup_{0\leqslant\tau\leqslant s}\|w_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|\nabla_v Df(\tau,\cdot)\|_p]\,ds \end{aligned}$$

($p=1$ or $p=\infty$);

(iii) $$\begin{aligned} &\sup_{0\leqslant s\leqslant t}\|\chi(s,\cdot)\|_\infty\\ &\quad\leqslant \int_0^t [\eta \sup_{0\leqslant\tau\leqslant s}\|E[\tilde{\rho}](\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|\chi(\tau,\cdot)\|_\infty\\ &\qquad+\sup_{0\leqslant\tau\leqslant s}\|w_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant\tau\leqslant s}\|(1+|v|^2)^{\eta/2}\,\nabla_v f(\tau,\cdot)\|_\infty]\,ds \end{aligned}$$

(iv) $\sup_{0\leqslant s\leqslant t} \|\xi_\Delta(s,\cdot)\|_\infty$

$$\leqslant \int_0^t \{[\eta \sup_{0\leqslant \tau\leqslant s} \|E[\tilde{\rho}](\tau,\cdot)\|_\infty + \sup_{0\leqslant \tau\leqslant s} \|\tilde{A}(\tau,\cdot)\|_\infty] \sup_{0\leqslant \tau\leqslant s} \|\xi_\Delta(\tau,\cdot)\|_\infty + \sup_{0\leqslant \tau\leqslant s} \|Dw_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant \tau\leqslant s} \|(1+|v|^2)^{\eta/2}\nabla_v f(\tau,\cdot)\|_\infty + \sup_{0\leqslant \tau\leqslant s} \|w_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant \tau\leqslant s} \|(1+|v|^2)^{\eta/2} D\nabla_v f(\tau,\cdot)\|_\infty\}\, ds$$

(from (5.5)). Finally, we must control the growth of a Hölder semi-norm. We provide the inequality for $|D\Delta(t,\cdot)|_{\alpha,2N}$. The one for $|\xi_\Delta(s,\cdot)|_{\alpha,2N}$ is similar.

We have to follow the methodology from Section 3, which takes into consideration the Hölder continuity of the trajectories with respect to the phase space variables (x, v). From (5.3), and with the help of (3.13), we find

$$\begin{aligned}\sup_{0\leqslant s\leqslant t} |D\Delta(s,\cdot)|_{\alpha,2N} \leqslant \int_0^t [&\sup_{0\leqslant \tau\leqslant s} |\tilde{A}(\tau,\cdot)|_{\alpha,N} \sup_{0\leqslant \tau\leqslant s} \|D\Delta(\tau,\cdot)\|_\infty \\ &+ \sup_{0\leqslant \tau\leqslant s} \|\tilde{A}(\tau,\cdot)\|_\infty \sup_{0\leqslant \tau\leqslant s} |D\Delta(\tau,\cdot)|_{\alpha,2N} \\ &+ \sup_{0\leqslant \tau\leqslant s} |Dw_\Delta(\tau,\cdot)|_{\alpha,N} \sup_{0\leqslant \tau\leqslant s} \|\nabla_v f(\tau,\cdot)\|_\infty \\ &+ \sup_{0\leqslant \tau\leqslant s} \|Dw_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant \tau\leqslant s} |\nabla_v f(\tau,\cdot)|_{\alpha,2N} \\ &+ \sup_{0\leqslant \tau\leqslant s} |w_\Delta(\tau,\cdot)|_{\alpha,N} \sup_{0\leqslant \tau\leqslant s} \|\nabla_v Df(\tau,\cdot)\|_\infty \\ &+ \sup_{0\leqslant \tau\leqslant s} \|w_\Delta(\tau,\cdot)\|_\infty \sup_{0\leqslant \tau\leqslant s} |\nabla_v Df(\tau,\cdot)|_{\alpha,2N}]\, ds \\ &\times \exp(\alpha t \sup_{0\leqslant \sigma\leqslant t} \max\{\|\tilde{A}(\sigma,\cdot)\|_\infty, \|A(\sigma,\cdot)\|_\infty\})\end{aligned}$$

From these estimates, it follows similarly to the reasoning in Section 3 that $Y_\Delta(t)$ satisfies an estimate

$$Y_\Delta(t) \leqslant C \int_0^t \tilde{G}(Y_\Delta(\tau))\, d\tau$$

with a smooth monotone increasing nonnegative function $\tilde{G}$ such that $\tilde{G}(0)=0$. It follows that $0 \leqslant Y_\Delta(t) \leqslant \tilde{\zeta}(t)$, where $\tilde{\zeta}$ is the solution of the initial value problem

$$\zeta(0)=0, \quad \zeta'=C\tilde{G}(\zeta).$$

But the smoothness of $\tilde{G}$ implies that $\tilde{\zeta} \equiv 0$, hence $Y_\Delta(t)=0$ on its interval of existence. Uniqueness follows. ∎

APPENDIX. PROOFS OF THE ESTIMATES FROM SECTION 2

As before, $f(t, x, v)$ denotes a density distibution function, N is the dimension of physical space (typically $N=3$), $\rho(t, x)=\int f(t, x, v)\, dv$, and w is as defined in Section 2. We also abbreviate $w_1(t, x)=E_1[\rho](t, x)$ and $w_2(t, x)=E_2[\rho](t, x)$. We will suppress the dependence on t in order to simplify the notation. All norms are with respect to the spatial variables. By X we denote the space

$$X=L^1 \cap L^\infty \cap H^\alpha$$

with norm $\|\cdot\|_X=\|\cdot\|_1+\|\cdot\|_\infty+|\cdot|_\alpha$.

A. *Proof of (2.1.).* Estimate

$$\begin{aligned}
|w_1(x)| &\leqslant |\gamma(N-2)| \int \frac{1}{|x-y|^{N-1}}\, \rho(y)\, dy \\
&= |\gamma|(N-2)\left\{\int_{|y-x| \leqslant R} \cdots dy+\int_{|y-x|>R} \cdots dy\right\} \\
&\leqslant |\gamma|(N-2)\{\|\rho\|_\infty\, \omega_N R+R^{-N+1} \omega_N\, \|\rho\|_1\}
\end{aligned}$$

By choosing $R=\|\rho\|_\infty^{-1/N}\, \|\rho\|_1^{1/N}$ (which minimizes the r.h.s. as a function of R), the first part of the estimate (2.1) follows. Repeating the same steps for w_2 as given by (1.3) completes the proof.

B. *Proof of (2.2.).* Write $Dw=Dw_1+Dw_2$, and estimate the two terms separately. As above, we break up the integrals defining Dw_1 (in this calculation, think of Dw_1 as any partial derivative of any component of w_1):

$$\begin{aligned}
Dw_1 &= \gamma(N-2) \int |x-y|^{-N}\, (x-y)\, D\rho(y)\, dy \\
&= \gamma(N-2)\left\{\int_{|x-y| \geqslant d_1} \cdots dy+\int_{|x-y|<d_1} \cdots dy\right\} \\
&:= J_1(x)+J_2(x)
\end{aligned}$$

J_2 is estimated, as in the previous proof, by

$$|J_2(x)| \leqslant |\gamma|(N-2)\,\omega_N d_1\,\|D\rho\|_\infty$$

As for J_1, we integrate by parts (i.e., use the divergence theorem) and find

$$\begin{aligned}|J_1(x)| \leqslant C\,|\gamma|(N-2)&\int_{|x-y|\geqslant d_1}\left(\nabla_y\cdot\frac{x-y}{|x-y|^N}\right)\rho(y)\,dy\\ &+|\gamma|(N-2)\int_{|x-y|=d_1}\frac{1}{|x-y|^{N-1}}\,\rho(y)\,d\sigma(y)\end{aligned}$$

and by using that $|\partial_{x_i}((x-y)/|x-y|^N)| \leqslant N\,|x-y|^{-N}$, we have

$$|J_1(x)| \leqslant C\,|\gamma|\,N(N-2)\int_{|x-y|\geqslant d_1}\frac{1}{|x-y|^N}\,\rho(y)\,dy + |\gamma|(N-2)\,\omega_N\,\|\rho\|_\infty$$

The integral in the first term on the right is further decomposed as

$$\begin{aligned}&\int_{d_2\geqslant|x-y|\geqslant d_1}\frac{1}{|x-y|^N}\,\rho(y)\,dy+\int_{|x-y|\geqslant d_2}\frac{1}{|x-y|^N}\,\rho(y)\,dy\\ &\quad\leqslant \omega_N\ln(d_2/d_1)\|\rho\|_\infty + d_2^{-N}\,\|\rho\|_1\end{aligned}$$

Collecting terms, we find that for all $d_2 > d_1 > 0$,

$$\begin{aligned}\|Dw_1\|_\infty &\leqslant |\gamma|(N-2)\,\omega_N\,d_1\,\|D\rho\|_\infty + C\,|\gamma|\,N(N-2)\,d_2^{-N}\,\|\rho\|_1\\ &\quad+|\gamma|(N-2)\,\omega_N\,\|\rho\|_\infty + C\,|\gamma|\,N(N-2)\,\omega_N\ln(d_2/d_1)\|\rho\|_\infty\\ &=|\gamma|(N-2)\,\omega_N\,\|\rho\|_\infty\,[1+CN\ln(d_2/d_1)]\\ &\quad+C\,|\gamma|\,N(N-2)\,d_2^{-N}\,\|\rho\|_1+|\gamma|\,\omega_N(N-2)\,d_1\,\|D\rho\|_\infty\end{aligned}$$

Now let $d_2 = 1$ and $d_1 = (1+\|D\rho\|_\infty)^{-1}$, then

$$\begin{aligned}\|Dw_1\|_\infty \leqslant C(\gamma, N)[\,&\|\rho\|_\infty(1+\ln(1+\|D\rho\|_\infty))\\ &+\|\rho\|_1+\|D\rho\|_\infty\,(1+\|D\rho\|_\infty)^{-1}]\end{aligned}$$

This completes the estimate for Dw_1.

As for the "Manev" term Dw_2, the $p=\infty$ modification of the classical Calderon and Zygmund argument (see [St] and [Ta]) entails that we have an estimate

$$|\partial_{x_i} w_2| \leqslant C_2(\varepsilon, N, \alpha)\|\nabla\rho\|_X$$

Combining the estimates for Dw_1 and Dw_2 and observing that $(1+\|\nabla\rho\|)^{-1} \leqslant 1$, we have

$$\begin{aligned}\|Dw\|_\infty &\leqslant \|Dw_1\|_\infty + \|Dw_2\|_\infty\\ &\leqslant C_1(\gamma, N)[\,\|\rho\|_\infty (1+\ln(1+\|\nabla\rho\|_\infty)) + \|\rho\|_1 + \|\nabla\rho\|_\infty]\\ &\quad + C_2(\varepsilon, N, \alpha)[\,\|\nabla\rho\|_1 + \|\nabla\rho\|_\infty + |\nabla\rho|_\alpha]\end{aligned}$$

This completes the proof of (2.2).

C. *Proofs of (2.3) and (2.4).* These estimates are consequences of Hörmander's Lemma (see [H]) which we state here for the sake of completeness.

Hörmander's Lemma [H]. Assume that the convolution kernel $k \in C^\infty(\mathbb{R}^N - \{0\})$ is homogeneous of degree $-N/\beta$, and suppose that $p \in [1, \infty]$ is such that $\xi := N(1-1/\beta-1/p) \in (0, 1)$. Then

$$\sup_{x \neq y} |k * u(x) - k * u(y)| \cdot |x-y|^{-\xi} \leqslant C(N/\beta, p)\|u\|_p$$

To prove (2.3), set $\beta = N/(N-1)$; then $\xi = 1-(N/p) \in (0, 1)$ if $p > N$, and Hörmander's Lemma entails that

$$\begin{aligned}|w|_\alpha &\leqslant |w_1|_\alpha + |w_2|_\alpha\\ &= \gamma(N-2)\,\big||x|^{-N} x * \rho\big|_\alpha + \varepsilon\,\big||x|^{-N+1} * \rho\big|_\alpha\\ &\leqslant \gamma(N-2)\; C_1(N, \alpha)\|\rho\|_p + \varepsilon C_2(N, \alpha)\|\rho\|_p\end{aligned}$$

(2.3) follows immediately.

For the proof of (2.4), consider $|Dw|_\alpha$. As before, write $w = w_1 + w_2$, where Dw_1 is represented by

$$Dw_1 = -\gamma(N-2)\int |x-y|^{-N}\,(x-y)\;\nabla_y\rho\; dy$$

Hence, by Hörmander's Lemma,

$$|Dw_1|_\alpha \leqslant C(N, \alpha, \gamma)\|\nabla\rho\|_p$$

Let $Tf(x) = \int |x-y|^{-N-1}(x-y)\, f(y)\, dy$, and let S_x denote the shift operator by x. Following the $p = \infty$ Calderon–Zygmund argument for w_2, we have

$$|Dw_2(t,x) - Dw_2(t,y)| = C(N,\varepsilon)\, |(S_{x-y} - I)\, T\nabla\rho(t,y)|$$

It is known [Ta] that $|T\nabla\rho|_\alpha$ is bounded in terms of $|\nabla\rho|_\alpha$ (for $0 < \alpha < 1$). Therefore,

$$|(S_{x-y} - I)\, T\nabla\rho(t,y)| \leqslant C(N,\varepsilon,\alpha) \sup_h |h|^{-\alpha}\, \|(S_h - I)\, \nabla\rho(t,\cdot)\|_\infty\, |x-y|^\alpha$$

and

$$|Dw_2|_\alpha \leqslant C(N,\varepsilon,\alpha)\, |\nabla\rho|_\alpha$$

readily follows. This completes the proof of (2.4).

It remains to sketch proofs for the interpolation estimates (2.5) to (2.7). For (2.5), an integral decomposition works as follows (we will continue to suppress the time variable):

$$\begin{aligned} |\rho(x)| &= \left| \int f(x,v)\, dv \right| \\ &= \left| \int_{|v| \leqslant R} \cdots + \int_{|v| \geqslant R} \cdots \right| \\ &\leqslant \omega_N \frac{R^N}{N} \|f\|_\infty + \|(1+|v|^2)^{\eta/2} f(\cdot,\cdot)\|_\infty \int_{|v|>R} (1+|v|^2|^{-\eta/2}\, dv \\ &= \omega_N \frac{R^N}{N} \|f\|_\infty + \omega_N R^{N-\eta} \frac{1}{\eta - N} \|(1+|v|^2)^{\eta/2} f\|_\infty \end{aligned}$$

(2.5) follows by choosing $R = \|(1+|v|^2)^{\eta/2} f\|_\infty^{1/\eta}\, \|f\|_\infty^{-1/\eta}$. Exactly the same argument applies for (2.6).

To prove (2.7), we apply the above estimate to $(S_h - I)\, \nabla\rho$:

$$\begin{aligned} &\|(S_h - I)\, \nabla\rho\|_\infty \\ &\quad \leqslant \eta \frac{\omega_N}{N(\eta - N)} \left(\|(S_h - I)\, \nabla f\|_\infty^{1-(N/\eta)}\, \|(S_h - I)(1+|v|^2)^{\eta/2}\, \nabla f\|_\infty^{N/\eta} \right) \end{aligned}$$

Estimate (2.7) then follows after multiplying both sides by $|h|^{-\alpha}$ and taking suprema with respect to h.

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REFERENCES

[B] J. Batt, Global symmetric solutions of the initial value problem of Stellar dynamics, *J. Differ. Equations* **25**:342–364 (1977).

[BDIV] A. V. Bobylev, P. Dukes, R. Illner, and H. D. Victory, On Vlasov–Manev equations, I: Foundations, properties and non-global existence, *J. Stat. Phys.* **88**:885–912 (1997).

[H] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, I. Theorem 4.5.10, p. 122, Springer-Verlag.

[K] R. Kurth, Das Anfangswertproblem der Stellardynamik, *Z. Astrophysik* **30**:213–229 (1952).

[Ma1] G. Manev, La gravitation et le principe de l'égalité de l'action et de la réaction, *Comptes Rendues* **178**:2159–2161 (1924).

[Ma2] G. Manev, Die Gravitation und das Prinzip von Wirkung und Gegenwirkung, *Z. Physik* **31**:786–802 (1925).

[Ma3] G. Manev, Le principe de la moindre action et la gravitation, *Comptes Rendues* **190**:963–965 (1930).

[Ma4] G. Manev, La gravitation et l'énergie au zéro, *Comptes Rendues* **190**:1374–1377 (1930).

[St] E. M. Stein, Singular integrals and differentiability properties of functions (Princeton University Press, 1970).

[Ta] M. H. Taibleson, The preservation of Lipschitz spaces under singular integral operators, *Studia Math.* **XXIV** (1964)